We discuss concept lattice in rough sets, define rough formal context, rough concept, and we give their properties. Furthermore, we investigate rough Galois connection and study the lower (upper) approximate operators by means of implication in rough formal context.

Keywords: Rough concept; rough Galois Connection; lower approximation; upper approximation.

1. Introduction

In 1982, R.Wille proposed a new model to represent the formal concepts associated to a context \((G, M, I)\), named formal concept analysis based on formal context, which is a binary relation between a set of objects and a set of attributes. The main goal is to reveal the hierarchical structure of formal concepts and to investigate the dependencies among attributes. The family of all formal concepts is a complete lattice, which is an effective method for several real-world applications in data analysis, such as object-oriented databases, inheritance lattices, mining for association rules, generating frequent sets etc. One of the important challenges in data handling is generating or navigating the concept lattice of binary relation. The theory of rough sets, proposed by Z.Pawlak, is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. The knowledge about a considered universe is the starting point. Using two operations, a lower approximation and an upper approximation, we can describe every subset of the universe. The concepts of the lower and upper approximations in rough set theory are fundamental to the examination of granularity in knowledge. These concepts is an effective way of studying imprecision, vagueness, and uncertainty. In this paper, we discuss the rough properties of concept lattice in rough set.

The rest of the paper is organized as following, in section 2, we define rough
formal context, rough concept, and we give their properties. In section 3, we study rough Galois connection. Conclusions are given in section 4.

2. Rough Concept Lattice

**Definition 2.1** Let \((G, M, F, R)\) be an information system, where \(G = \{a_1, \ldots, a_m\}\) is an object set, \(M = \{x_1, \ldots, x_n\}\) is an attribute (property) set, \(R\) is an equivalent relation on \(G\), \(\forall A \subseteq G\), we can define the upper and the lower approximation of \(A\) about \(R\):

\[
\overline{A}_R = \bigcup \{Y \in G/R \mid Y \cap A \neq \emptyset\} = \{x \in G \mid [x]_R \cap A \neq \emptyset\};
\]

\[
\underline{A}_R = \bigcup \{Y \in G/R \mid Y \subseteq A\} = \{x \in G \mid [x]_R \subseteq A\}.
\]

\(\overline{A}, \underline{A}\) are called \(R\)– upper approximation and \(R\)– lower approximation of \(A\). \(\overline{A}\) and \(\underline{A}\) are denoted simply respectively. If \(\overline{A} = \underline{A}\) we say that \(A\) is definable, otherwise, \(A\) is rough. Similarly, we can define the upper and lower approximation of attributes set \(B \subseteq M\) about an equivalent relation on \(M\).

Yao discussed the (mutual) definability of sets of objects and properties by the conjunctive and disjunctive operators in \(^8\), we can classify the sets of objects and properties by equivalent relation, so we have:

**Definition 2.2** \(\forall B \subseteq M\), we denote \(R_B = \{(a_i, a_j) \in G \times G \mid f_i(a_i) = f_j(a_j), l \in B\}\), where \(f_i : G \rightarrow \{0, 1\}\) is defined by \(f_i(a_i) = 1\) if and only if the object \(a_i\) possesses property \(l\), \((l \in M, a_i \in G)\), and \(R_B\) is an equivalent relation, and \(R_B\) can generate a partition of \(G\), \(A(B) = \{[x]_B \mid x \in G\} = G/R_B\), where \([x]_B = \{y \in G \mid yR_x \supseteq x\} = \{y \in G \mid f_i(y) = f_i(x), a_i \in B\}\), that is, \([x]_B = \bigwedge\{[a_i, f_i(x)] \mid a_i \in B\}\).

**Example 1** The following table is rough formal context \((G, M, F, R)\), where \(G = \{1, \ldots, 5\}, M = \{a, \ldots, i\}\).

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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</tr>
</tbody>
</table>

By rough theory, \(A = \{1, 2\} \subseteq G, B = \{a, i\} \subseteq M\), then \(f_a(1) = f_a(2) = f_a(3) = 1, f_a(4) = f_a(5) = 0, f_b(1) = f_b(2) = f_b(3) = 0, f_b(4) = f_b(5) = 1, f_c(1) = f_c(2) = f_c(4) = 1, f_c(3) = f_c(5) = 0, \cdots\); and the partition of \(A\) is : \(M/[1]_A = \{a, c, f, h\}, \{b, d, e, g, i\}\), \(M/[2]_A = \{a, c, g\}, \{b, d, e, f, h\}\), so, \(M/A = \{\{a, c\}, \{f, h\}, \{g, i\}, \{d, e\}\}\); and the partition of \(B\) is \(G/[a]_B = \)
\[\{1,2,3\}, \{4,5\}\}, G/\{i\}_B = \{\{2,3\}, \{1,4,5\}\}\), so, \(G/B = \{\{1\}, \{2,3\}, \{4,5\}\}\). Under \(M/A = R_A\), the lower approximation of \(B = \overline{B}_{R_A} = \emptyset\), the upper approximation of \(B = \overline{B}_{R_A} = \{a,c,g,i\}\), and under \(M/B = R_B\), the lower approximation of \(A = \overline{A}_{R_B} = \{1\}\), the upper approximation of \(A = \overline{A}_{R_B} = \{1,2,3\}\).

In fact, \(f_1 : G \rightarrow \{0,1\}\), set \(\{0,1\}\) can extent to \([0,1]\), and the condition of equivalent relation can also substitute for other relations, for example, the similar those functions are the same.

The information system \((G, M, F, R)\) which has lower, upper approximation and partition is called a rough formal context.

\[\forall g \in G, m \in M, \text{ object } g \text{ has attribute } m, \text{ then } (g, m) \in F, \text{ or } gFm.\]

**Definition 2.3** Let \((G, M, F, R)\) is a rough formal context, \(F \subseteq G \times M\), for a set \(B \subseteq M\) of attributes , we define function \(1 : 2^M \rightarrow 2^U\), \(B^1 = \{g \in G \mid (g, m) \in R, \forall m \in B\}\) (the set objects which have all attributes in \(B\)). Correspondingly, for a set \(A \subseteq G\) of objects ,we define: \(1 : 2^U \rightarrow 2^M\), as following:

\[A^1 = \{m \in M \mid (g, m) \in F, \forall g \in A\}\] (the set of attributes common to the objects of in \(A\)),

and its negation:

\[B^1 = \{g \in G \mid (g, m) \notin F, \forall m \in B\}\] (the set objects which have not all attributes in \(B\)).

\[A^* = \{m \in M \mid (g, m) \notin F, \forall g \in A\}\] (the set of attributes common not to the objects of in \(A\)).

By the definition, \(\forall x \in G, \{x\}^1 = \{y \in M \mid xFy\} \subseteq M\) is the set of attribute possessed by object \(x\), and \(\forall y \in M, \{y\}^1 = \{x \in G \mid xFy\} \subseteq G\) is the set of objects having attribute \(y\). For a set of objects \(A\), \(A^1\) is the maximal set of attributes shared by all objects in \(A\), for a set of attributes \(B\), \(B^1\) is the maximal set of objects that have all attributes in \(B\), their negation is similar.

**Example 2** Consider example 1, the following figure is its rough concept lattice, in here, \(F = \{0,1\}\), that is , we only consider \(\{g, m\} \in F\) or \(\{g, m\} \notin F\).

![Fig. 1. Rough concept lattice](image-url)
Definition 2.4 A rough formal concept of a rough formal context \((G, M, F, R)\) is a pair of \((A, B)\) with \(A \subseteq G, B \subseteq M, B^\uparrow = A, A^\uparrow = B\). We call \(A\) the extent and \(B\) the intent of rough concept \((A, B)\). \(B(G, M, F, R) \subseteq 2^G \times 2^M\) denotes the set of all rough concepts of rough formal context \((G, M, F, R)\), which is a complete lattice with order relation \((A, B) \leq (C, D) \iff A \subseteq C \iff D \subseteq B\).

Proposition 2.1 If \((G, M, F, R)\) is a rough formal context, \(A, A_1, A_2 \subseteq G\) are sets of objects and \(B, B_1, B_2\) are sets of attributes, \(F \subseteq G \times M\) is a rough relation between \(G\) and \(M\). then

1. \(A_1 \subseteq A_2 \Rightarrow A_2 \subseteq A_1, B_1 \subseteq B_2 \Rightarrow B_2 \subseteq B_1\);
2. \(A \subseteq A^\uparrow, B \subseteq B^\uparrow\);
3. \(A = A^\uparrow, B = B^\uparrow\);
4. \(A \subseteq B \iff B \subseteq A\).

Obviously, \(1 : 2^M \to 2^U, \uparrow : 2^U \to 2^M\), that is \((1, \uparrow)\) forms Galois connection, and calls the rough Galois connection.

Proposition 2.2 A pair \((\uparrow, \downarrow)\) is a rough Galois connection if and only if \(A \subseteq B^\uparrow \iff B \subseteq A^\uparrow\), where \(A \subseteq G, B \subseteq M\).

Proof By proposition 2.1, \(A \subseteq B^\uparrow \Rightarrow B \subseteq A^\uparrow\), and \(B \subseteq B^\uparrow\), hence, \(B \subseteq A^\uparrow\), i.e. one direction holds, the other follows symmetrically.

Theorem 2.1 The rough formal concept lattice \(L = B(G, M, F, R)\) is a complete in which the infimum and supremum are given by

\[
\bigwedge_{t \in T}(A_t, B_t) = (\bigcap_{t \in T} A_t, (\bigcup_{t \in T} B_t)^\uparrow)
\]

\[
\bigvee_{t \in T}(A_t, B_t) = ((\bigcup_{t \in T} A_t)^\uparrow, \bigcap_{t \in T} B_t)
\]

where \(T\) is a index set and for every \(t \in T\), \(A_t \subseteq G, B_t \subseteq M, (A_t, B_t)\) is a rough formal concept.

Furthermore, a complete lattice \(V = (V, \leq)\) is isomorphic to the rough formal concept lattice \(L = B(G, M, F, R)\) if and only if there are mappings \(\gamma : G \leftarrow V\) and \(\mu : M \rightarrow V\) such that \(\gamma(G)\) is supremum-dense in \(V\), \(\mu(M)\) is infimum-dense in \(V\), and \(gFm\) is equivalent to \(\gamma \leq \mu\), for all \(g \in G\) and all \(m \in M\). In particular, \(V \cong B(V, V, F, R)\).

Proof We will explain the formula for the infimum. Since \(A_t = B_t^\uparrow, \) for each \(t \in T\), by proposition 2.1, having \((\bigcap_{t \in T} A_t, (\bigcup_{t \in T} B_t)^\uparrow) = ((\bigcap_{t \in T} B_t)^\uparrow, (\bigcup_{t \in T} B_t)^\uparrow)\), i.e. it has the form \((X^\uparrow, X^\downarrow)\) and is therefore certainly a rough concept, that this can only be the infimum, i.e. the largest common sub-concept of concept \((A_t, B_t)\). The formula for the supremum is substantiated correspondingly. Thus, we have proven that \(B(G, M, F, R)\) is a complete lattice.

To prove \(V \cong B(G, M, F, R)\), we can use the ideas proposed in [1]. In this paper we omit the details.
Proposition 2.3 Let \((G, M, F, R)\) is a rough formal context, \(A, B \subseteq G\), \(R\)– upper approximation and \(R\)– lower approximation of \(A\) have many properties:

(4) \(\overline{A} \subseteq A \subseteq \overline{A}\);
(5) \(\overline{A} \cup B = \overline{A \cup B}, \overline{A \cap B} = \overline{A} \cap \overline{B}\);
(6) \(\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}, \overline{A \cup B} \supseteq \overline{A} \cup \overline{B}\);
(7) \(\sim \overline{A} = \sim \overline{A} = \sim A\);
(8) \((\overline{A}) = \overline{(A)} = \overline{A}\);
(9) \(\overline{U} = U, \overline{\emptyset} = \emptyset\);
(10) if \(A \subseteq B\), then \(A \subseteq B, \overline{A} \subseteq \overline{B}\).

Proposition 2.4 Let \((G, M, F, R)\) is a rough formal context, \(B \subseteq M\), then the equivalent relation \(R_B\) and equivalent class \([x]_B\) have properties:

(11) If \(B_1 \subseteq B_2\), then \(R_{B_2} \subseteq R_{B_1}\), especially, if \(a \in B\), then \(R_M \subseteq R_B \subseteq R_a\);
(12) \(\forall B \subseteq M, R_B = \bigcap_{a \in B} R_a\);
(13) If \(B_1 \subseteq B_2 \subseteq M\), then \([x]_{B_2} \subseteq [x]_{B_1}\), especially, if \(a \in B\), then \([x]_M \subseteq [x]_{B} \subseteq [x]_a\);
(14) \(\forall B \subseteq M, [x]_B = \bigcap_{a \in B}[x]_a\).

Furthermore, we can define rough concept by the lower and upper approximate operators, that is, if \((A, B)\) is a rough concept, then \((\overline{A}, \overline{B}), (\overline{A}, \overline{B})\) are the lower and upper approximate rough concept correspondingly, they satisfy:

\[\overline{A} = \overline{B}, \overline{A} = \overline{B}, \overline{A} = \overline{B}\]

For rough set \(A \subseteq G\), the rough degree( or rough value) \(\rho_R(A)\) of \(A\) is called the membership degree of \(x\) in \(A\), and it is interpreted as the rough degree of “\(x\) is element of the lower, upper approximation of \(A\) about \((G, M, F, R)\)”, denotes \(\rho_R(A) = 1 - \overline{A}\), and \(\eta_R(\overline{A}) = \overline{\overline{A}}\) is the approximate accuracy of \(A\) about \((G, M, F, R)\).

The membership functions of \(A \subseteq G\) are defined as \(\overline{A}_R(x) = \inf\{A(y) | y \in [x]_R\}\); \(\underline{A}_R(x) = \sup\{A(y) | y \in [x]_R\}\), \(x \in G\), the membership function of \(B \subseteq M\) can define analogously. Such that, we can discuss rough degree of sets of attributes and objects in rough contexts.

In [8], Yao defines the lower and upper approximate operators by means of necessity and possibility operators, and a pair of set-theoretic operators\((\ast, \#)\), which called the sufficiency and the dual sufficiency operators. We can prove that though they have different formal, they are same in essential. Moreover, let \((G, M, F, R)\) be a rough formal context, by theorem 2.2, we know \(B(G, M, F, R)\) is a complete lattice, if we consider it as a residual lattice, \(F\) is a rough relation between \(G\) and \(M\), i.e. \(F \subseteq G \times M\), then we can also define rough lower and upper approximation rough concept by implication operators. For example, \(a \rightarrow b = \min(1 - a + b, 1)\) (Łukasiewicz implication); \(a \rightarrow b = 1\), if \(a \leq b\), otherwise \(a \rightarrow b = b\) (Gödel implication); \(a \rightarrow b = 1\), if \(a \leq b\), otherwise \(a \rightarrow b = (1 - a) \lor b\) (R₀ implication).

We can further consider a rough set \(F\) in a universe set \(X\) is any mapping \(A : X \rightarrow F(i.e.A \in X^F)\), where \(F\) is a suitable set of rough value, we can de-
Formal context is a classical set equal 1. It is well-defined, and this idea is in accordance with the graded rough set, the variable precision rough set model, and so on, for details see [7, 8]. Conclude that we can discuss the information system, machine learning, and specialist system by rough set. The value $A(x)$ is understood as the rough value (degree of truth) of "$x$ belong to $A". Moreover, a binary rough relation $R'$ between $G$ and $M$ is a mapping $R : G \times M \rightarrow F$, $R'(g, m)$ is interpreted as the rough value of "object $g$ has attribute $m$". For $A_1, A_2 \in X^F$, we put $A_1 \subseteq A_2 \iff \underline{A}_1(x) \leq \underline{A}_2(x) \iff \overline{A}_1(x) \leq \overline{A}_2(x)$ holds for all $x \in X$.

Let $R'$ be a binary rough relation between the objects set $G$ and attributes set $M$, i.e. $R' \subseteq G \times M$. For all $A \subseteq G$, define rough set $\overline{A}^{R'} \cup \underline{A}^{R'}$, simply denote $\overline{A}$ (dualy $\underline{A}$) in $M$ by:

$$\overline{A}(m) = \bigwedge_{g \in G} (A(g) \rightarrow R'(g, m)), \underline{A}(m) = \bigwedge_{g \in G} (A(g) \rightarrow R'(g, m)).$$

Similarly, for $B \subseteq M$, put $\overline{B}(g) = \bigwedge_{m \in M} (B(m) \rightarrow R'(g, m)), \underline{B}(g) = \bigwedge_{m \in M} (B(m) \rightarrow R'(g, m)).$

Such $\overline{A}(m)$ is the truth value of $\forall g \in A \Rightarrow (g, m) \in R'$, i.e. the rough value of "each objects of $A$ has attribute $m$", and analogously for $\underline{A}(m), \overline{B}(g), \underline{B}(g)$. So, given $G, M, R$, a rough concept is a pair $<\overline{A}, \underline{A}> (or <\overline{A}, \overline{B}>)$ in $L$, such that $\overline{A} \cup \underline{A} = \overline{B}, \overline{A} \cap \overline{B} = \overline{A}, \overline{B} \cap \overline{B} = \overline{A}$.

The set of all rough concepts will be denoted by $B(G, M, F, R)$. A order $\leq$ on $B(G, M, F, R)$ given by: $(A_1, B_1) \leq (A_2, B_2) \iff (A_1 \subseteq A_2) \iff (B_2 \subseteq B_1)$

$$\begin{array}{c c c c}
A_1 & A_2 & A_3 & A_4 & A_5 \\
1 & 0 & 0 & 1 & 1/3 \\
0 & 1/3 & 0 & 1/3 & 0 \\
\end{array}$$

**Example 3** The following table is rough formal context $(G, M, F, R)$, where $G = \{g_1, g_2\}, M = \{m_1, m_2, m_3\}, F = \{0, 1/3, 1\}$.

<table>
<thead>
<tr>
<th>No.</th>
<th>extent $[g_1, g_2]$</th>
<th>intent $[m_1, m_2, m_3]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${0, 0}$</td>
<td>${1, 1, 1}$</td>
</tr>
<tr>
<td>2</td>
<td>${1/3, 0}$</td>
<td>${2/3, 2/3, 1}$</td>
</tr>
<tr>
<td>3</td>
<td>${1, 0}$</td>
<td>${1, 0, 1/3}$</td>
</tr>
<tr>
<td>4</td>
<td>${1, 1/3}$</td>
<td>${2/3, 0, 1/3}$</td>
</tr>
<tr>
<td>5</td>
<td>${1, 1}$</td>
<td>${0, 0, 0}$</td>
</tr>
</tbody>
</table>
If we consider implication \( \rightarrow \) being Gödel operators, then table 4 is rough concept lattice of Gödel operators.

Table 4. Rough concept lattice (Gödel operators)

<table>
<thead>
<tr>
<th>No.</th>
<th>extent ( {g_1, g_2} )</th>
<th>intent ( {m_1, m_2, m_3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( {0, 0} )</td>
<td>( {1, 1, 1} )</td>
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<tr>
<td>2</td>
<td>( {1/3, 0} )</td>
<td>( {0, 1, 0} )</td>
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<tr>
<td>3</td>
<td>( {1/3, 0} )</td>
<td>( {0, 0, 1} )</td>
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<tr>
<td>4</td>
<td>( {0, 1} )</td>
<td>( {0, 1/3, 0} )</td>
</tr>
<tr>
<td>5</td>
<td>( {1, 0} )</td>
<td>( {1, 0, 1/3} )</td>
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<tr>
<td>6</td>
<td>( {1, 1/3} )</td>
<td>( {0, 0, 1/3} )</td>
</tr>
<tr>
<td>7</td>
<td>( {1, 1} )</td>
<td>( {0, 0, 0} )</td>
</tr>
</tbody>
</table>

**Theorem 2.2** The set \( B(G, M, F, R) \) is under \( \leq \) a complete lattice, where the supremum and infimum of the upper approximation are given by

\[
\bigwedge_{t \in T} (A_t, B_t) = (\bigcap_{t \in T} A_t, (\bigcup_{t \in T} B_t)')
\]

\[
\bigvee_{t \in T} (A_t, B_t) = ((\bigcup_{t \in T} A_t)', \bigcap_{t \in T} B_t)
\]

For the lower approximation, we have similar results, that is:

\[
\bigwedge_{t \in T} (A_t, B_t) = (\bigcap_{t \in T} A_t, (\bigcup_{t \in T} B_t))
\]

\[
\bigvee_{t \in T} (A_t, B_t) = ((\bigcup_{t \in T} A_t)', \bigcap_{t \in T} B_t)
\]

It is similar to theorem 2.1, a complete lattice \( \mathcal{V} = (V, \leq) \) is isomorphic to the
rough formal concept lattice $L = \mathcal{B}(G, M, F, R)$ if and only if there are mappings $\gamma : G \hookleftarrow V$ and $\mu : M \rightarrow V$ such that $\gamma(G)$ is supremum-dense in $V$, $\mu(M)$ is infimum-dense in $V$, and $gFm$ is equivalent to $\gamma \leq \mu$, for all $g \in G$ and all $m \in M$. In particular, $V \cong \mathcal{B}(V, V, F, R)$. The only of their difference is theorem 2.4 given the rough concepts with the lower and upper approximation.

The proof is similar to theorem 2.2. Therefore, we omit further explanation.

In [10], the authors gave the monotone concept (definition 10) and theorem 4.3, as following:

**Definition 10** A monotone concept in a monotone context $(G, M, F, R)$ is a pair $(A, F)$ where $A = \bigcup_{i=1}^{n} A_i$ is a monotone extent, $F = \bigvee_{i=1}^{n} B_i$ is a feasible monotone Boolean formula, $\delta(F) = A$, $\gamma(A) = F$, and each $(A_i, B'_i)$, where $B'_i$ is the set of features associated with $B_i$, is an elementary concept for $1 \leq i \leq n$.

In [10], the definition 5,6,7 gave the concept of a monotone formula, monotone context, and monotone extent, respectively.

**Theorem 4.3** Let $(G, M, F, R)$ be a monotone context, then $(\Phi(G, M, F, R), \leq)$ is a complete lattice in which supremum and infimum given by:

\[
\bigwedge_{j \in J} (A_j, F_j) = (\bigcup_{j \in J} A_j, \gamma(\bigcup_{j \in J} A_j))
\]

\[
\bigwedge_{j \in J} (A_j, F_j) = (\bigcap_{j \in J} A_j, \gamma(\bigcap_{j \in J} A_j))
\]

I think these definitions and theorem are not well-defined, because we all know the more extents possessed the less intents, as issues in [10] contradict with its, moreover, we can correct these, for example, in definition 5, we can change $F = \bigvee_{i=1}^{n} B_i$, $\delta(F) = \bigcup_{i=1}^{n} \delta(B_i)$ into $F = \bigwedge_{i=1}^{n} B_i$, $\delta(F) = \bigcap_{i=1}^{n} \delta(B_i)$.

3. **Rough Galois Connections**

In this section we study rough Galois connections between the sets of all rough sets in two given universes and their correspondence to binary relations, that is, discuss rough Galois connections in rough formal context $(G, M, F, R)$. We can study it in the residual lattice.

Suppose $L = \mathcal{B}(G, M, F, R)$ be a complete residual lattice. Given two rough sets $A_1, A_2 \in L^G$, we define the subsethood degree $\text{subs}(\overline{A_1}, \overline{A_2})$ of $A_1$ in $A_2$ by $\text{subs}(\overline{A_1}, \overline{A_2}) = \land_{x \in G} (\overline{A_1}(x) \rightarrow \overline{A_2}(x))$, or $\text{subs}(A_1, A_2) = \land_{x \in G} (\overline{A_1}(x) \rightarrow \overline{A_2}(x))$.

$\text{subs}(\overline{A_1}, \overline{A_2})$ is naturally interpreted as the rough degree of “for all $x \in G$ it holds that if $x \in \overline{A_1}$, then $x \in \overline{A_2}$”, and the lower approximation is similar.

In the following, we do not distinguish between the lower approximation and upper
approximate, because they are difference only existing their classification. As usual, by \( A_1 \subseteq A_2 \) for \( \text{subs}(A_1, A_2) = 1 \).

**Definition 3.1** A rough Galois connection (simply, \( R \)-Galois connection) between the set \( G \) and \( M \) is a pair \( <^{\downarrow},^{\uparrow}> \) of mapping \( ^{\downarrow} : \mathcal{L}^G \rightarrow \mathcal{L}^M, ^{\uparrow} : \mathcal{L}^B \rightarrow \mathcal{L}^G \) satisfying:

\[
\text{subs}(A_1, A_2) \leq \text{subs}(A_1', A_2'); \text{subs}(B_1, B_2) \leq \text{subs}(B_1', B_2'); A \subseteq (A')^\downarrow; B \subseteq (B')^\uparrow.
\]

We can consider \( \mathcal{L} \)-rough formal context which is defined as a tuple \((\mathcal{L}, G, M, F, R)\), where \( \mathcal{L} = \mathcal{B}(G, M, F, R) \) is a complete lattice with negation \( ^\prime \), \( t \)-co-norm \( \top \), \( G, M, F, R \) are same as before, such, we can define \( ^{\downarrow} : \mathcal{L}^M \rightarrow \mathcal{L}^{G^\downarrow}, ^{\uparrow} : \mathcal{L}^G \rightarrow \mathcal{L}^{M^\uparrow}:
\]

\[
\forall A \subseteq U, \overrightarrow{A}(m) = \inf_{m \in \mathcal{M}}(B(g))^\top R(g, m); \quad A^\downarrow(m) = \inf_{m \in \mathcal{M}}(B(g))^\top R(g, m).
\]

\[
\forall B \subseteq M, \overleftarrow{B}(g) = \inf_{g \in \mathcal{G}}(A(g))^\top R(g, m); \quad B^\uparrow(g) = \inf_{g \in \mathcal{G}}(A(g))^\top R(g, m)).
\]

Obviously, in this case \( <^{\downarrow},^{\uparrow}> \) is still rough Galois connection. The next theorem provides us with a simple characterization of \( R \)-Galois connections. Simply, we only denote \( A^\downarrow, B^\uparrow \) which include lower and upper approximation.

**Theorem 3.1** A pair \( <^{\downarrow},^{\uparrow}> \) forms a rough connection between \( G \) and \( M \) iff for all \( A \in \mathcal{L}^G, B \in \mathcal{L}^M, \text{subs}(A, B^\downarrow) = \text{subs}(B, A^\uparrow) \).

**Proof** Let \( <^{\downarrow},^{\uparrow}> \) be a rough Galois connection. Form \( B \subseteq (B^\uparrow)^\downarrow \), we can get \( \text{subs}(B^\uparrow)^\downarrow, A^\downarrow) \leq \text{subs}(B, A^\downarrow) \), hence, \( \text{subs}(A, B^\downarrow) \leq \text{subs}(B^\uparrow, A^\downarrow) \leq \text{subs}(B, A^\downarrow) \). Repeating the arguments we get \( \text{subs}(A, B^\downarrow) \geq \text{subs}(B, A^\downarrow) \) so \( \text{subs}(A, B^\downarrow) = \text{subs}(B, A^\downarrow) \).

Conversely, let \( \text{subs}(A, B^\downarrow) = \text{subs}(B, A^\downarrow) \). From \( \text{subs}(A^\downarrow, A^\downarrow) = 1 \) i.e. \( A \subseteq (A^\downarrow)^\downarrow \), \( B \subseteq (B^\uparrow)^\downarrow \) may be proved symmetrically. From \( A_2 \subseteq (A^\downarrow)^\downarrow \), it follows by \( \text{subs}(A, B^\downarrow) = \text{subs}(B, A^\downarrow) \) that \( \text{subs}(A_1, A_2) \leq \text{subs}(A_1, (A^\downarrow)^\downarrow) = \text{subs}(A_1', A_2') \), proving \( \text{subs}(A_1, A_2) \leq \text{subs}(A_2, A_2') \).

The proof of \( \text{subs}(B_1, B_2) \leq \text{subs}(B_2, B_1') \) is symmetric.

We can also use implication operator considering rough Galois connection, that is, we can let \( \top \) be an implication operator. Furthermore, we have:

**Definition 3.2** Let \( K \subseteq F \), a \( \mathcal{L} - K \) rough Galois connection between the set \( G \) and \( M \) is a pair \( <^{\downarrow},^{\uparrow}> \) of mapping \( ^{\downarrow} : \mathcal{L}^G \rightarrow \mathcal{L}^M, ^{\uparrow} : \mathcal{L}^B \rightarrow \mathcal{L}^G \) satisfying:

\( \forall A, A_1, A_2 \in \mathcal{L}^G, B, B_1, B_2 \in \mathcal{L}^M, \)

\[
\begin{align*}
(1) & \quad \text{subs}(A_1, A_2) \leq \text{subs}(A_1', A_2'); \quad \text{subs}(B_1, B_2) \leq \text{subs}(B_1', B_2'); \quad A \subseteq (A')^\downarrow; \quad B \subseteq (B')^\uparrow; \\
(2) & \quad \text{subs}(B_1, B_2) \leq \text{subs}(B_1', B_2'); \quad \text{subs}(B_1, B_2) \subseteq K;
\end{align*}
\]

**Remark 1** Note that the role ok \( K \) is to control the sensitivity of conditions “
the more objects, the less common attributes” and “the more attributes, the less common objects sharing them”. for \( K = \emptyset \), we have only the conditions (3) and (4), which say that the composed mappings \( {}^\updownarrow : \mathcal{L}^G \rightarrow \mathcal{L}^G, {}^\downarrow : \mathcal{L}^M \rightarrow \mathcal{L}^M \) are extensive.

Remark 2 Clearly, for \( K_1 \subseteq K_2 \), each \( \mathcal{L} - K_2 \) Galois connections is also \( \mathcal{L} - K_1 \) Galois connections.

Remark 3 If \( K = \{1\} \), then \( \mathcal{L} - K \) Galois connections degenerate rough Galois connections.

4. Conclusion

In this paper, we discuss concept lattices based on rough sets. We define the rough formal context, rough formal concept and rough Galois connections. Furthermore, we give their properties that offer a new method and tool in data analysis. Therefore, we can deal with lots of data more easily and do more efficient decision in data mining, information system, human reasoning and so on. Moreover, we can discuss the properties of former concepts, such as, the graded rough concept and similarity or class of rough concepts. These will be discussed in another article.

Acknowledgments

This work is partly supported by the Education department chunhui program(Z2006-1-81001).

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